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# Time-reversible Hamiltonian systems

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It is shown that transfer matrices satisfying  $G(-s) = G(s) = G^T(-s)$  have a minimal Hamiltonian realization with an energy which is the sum of potential and kinetic energy, yielding the time reversibility of the equations. Furthermore connections are made with an associated gradient system. The theory is applied to electrical networks, resulting in a characterization of LCT networks.

**Keywords:** Hamiltonian systems, Time reversibility, Realization theory, Electrical networks.

## 1. Introduction and statement of the results

As is well known (see for instance [3,6]),  $m \times m$  transfer matrices  $G(s)$  satisfying the symmetry property  $G(s) = G^T(-s)$  have a minimal realization of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

$x \in \mathbb{R}^{2n}$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^m$ , with  $\{A, B, C, D\}$  satisfying

$$A^T J + JA = 0, \quad B^T J = C, \quad D = D^T,$$

where  $J$  is a so-called symplectic form on  $\mathbb{R}^{2n}$ . By Darboux's theorem there exists a basis of  $\mathbb{R}^{2n}$  in which  $J$  has the matrix form

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Such transfer matrices and their realizations as above are called *Hamiltonian*. They arise in numerous places, e.g. mechanical systems, LC-circuits, spectral densities and optimization techniques.

On the other hand, *reversible* systems are characterized by the condition that if  $(u(t), y(t))$ ,  $t \in \mathbb{R}$ , belongs to the external behaviour of the system, then also the time-reversed signal  $(u(-t),$

$y(-t))$ ,  $t \in \mathbb{R}$ , is a feasible external behaviour. This is equivalent to the condition  $G(s) = G(-s)$ . In this case we can find a minimal realization

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

$x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^p$ , where (see [8])

$$A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B \end{pmatrix}, \quad C = (C_1 \quad 0)$$

with

$A_1$  a  $(k \times l)$ -matrix,

$A_2$  an  $((n-k) \times (n-l))$ -matrix,

$B_1$  an  $((n-k) \times m)$ -matrix

and

$C_1$  a  $(p \times (n-l))$ -matrix

$(k, l \leq n)$ .

Such a transfer matrix and its minimal realization as above are called *reversible*.

In this note we will study transfer matrices satisfying both conditions, i.e.  $G(s) = G^T(-s)$  and  $G(s) = G(-s)$ . We will show that we can construct a minimal realization which is Hamiltonian as well as reversible. Explicitly, there exists a realization

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

$x \in \mathbb{R}^{2n}$ ,  $u \in \mathbb{R}^m$ , which has the form

$$\dot{q} = Pp,$$

$$\dot{p} = -Qq + \tilde{B}u, \quad y = \tilde{C}q + Du,$$

with  $q \in \mathbb{R}^n$ ,  $p \in \mathbb{R}^n$ ,  $x = \begin{pmatrix} q \\ p \end{pmatrix}$  and  $\tilde{B}^T = \tilde{C}$ ,  $D = D^T$ ,  $P = P^T$  and  $Q = Q^T$ .

This represents a mechanical system in which the energy has the simple form  $\frac{1}{2}q^T Q q + \frac{1}{2}p^T P p$ , the first term denoting the potential energy, and the second term equal to the kinetic energy. Finally we mention that topological properties of exactly this class of transfer matrices have been studied in [4].

**Remark.**  $G(s) = G^T(-s)$  and  $G(s) = G(-s)$  implies  $G(s) = G^T(s)$ . Transfer matrices satisfying

the last condition also have a nice minimal realization, called a reciprocal or gradient system (see for instance [9]). One could wonder if it is also possible to construct a minimal realization which is both reciprocal and reversible. However it is easy to see that is *not* the case.

## 2. Theorems and proofs

**Theorem 1.** Let  $G(s)$  be a (proper)  $m \times m$  transfer matrix. Then  $G(s) = G^T(-s)$  and  $G(s) = G(-s)$  if and only if there exists a minimal realization

$$\dot{x} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \tilde{B} \end{pmatrix} u, \quad (2.1)$$

$$y = (\tilde{C} \ 0)x + Du,$$

$x \in \mathbb{R}^{2n}$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^m$ , with  $P$  and  $Q$  ( $n \times n$ )-matrices,  $\tilde{B}$  an ( $n \times m$ )-matrix,  $\tilde{C}$  an ( $m \times n$ )-matrix and  $D$  an ( $m \times m$ )-matrix, satisfying  $P = P^T$ ,  $Q = Q^T$ ,  $\tilde{B}^T = \tilde{C}$  and  $D = D^T$ .

Moreover the pair  $(PQ, P\tilde{B})$  is controllable (hence  $P$  has at least full rank).

**Proof.** All we have to do is to combine the state space isomorphism theorem (see for instance [3,9]) with [5, Lemma 1]. Let  $\{A, B, C, D\}$  be a minimal realization of  $G(s)$ . Because  $G(s) = G^T(-s)$ , also  $\{-A^T, -C^T, B^T, D^T\}$  is a minimal realization. Therefore there exists a unique bijective map  $J$  such that

$$-A^T = JAJ^{-1}, \quad -C^T = JB, \quad B^T = CJ^{-1}.$$

Since also  $-J^T$  satisfies the above equations it follows from the uniqueness of  $J$  that  $J = -J^T$ .

Because  $G(s) = C(-s)$ , also  $\{-A, -B, C, D\}$  is a minimal realization. Therefore there exists a map  $R$  such that

$$-A = RAR^{-1}, \quad -B = RB, \quad C = CR^{-1}.$$

Also  $R^{-1}$  satisfies these equations, hence  $R = R^{-1}$ .

From the identities above it easily follows that for  $r = 0, 1, 2, \dots$

$$R^T J (A^r B) = -J R (A^r B).$$

Therefore, by controllability of  $(A, B)$ ,  $R^T J = -J R$ , or  $R^T J R = -J$ , i.e.  $R$  is an *anti-symplectomorphism*. Now [5, Lemma 1] says that we can find a basis of the state space – because  $J = -J^T$  necessarily even-dimensional, hence  $\mathbb{R}^{2n}$

– such that in this basis

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Because  $\{A, B, C, D\}$  has to satisfy

$$\begin{aligned} A^T J + J A &= 0, & A &= -R A R, \\ B^T J &= C, & B &= -R B, \\ D &= D^T, & C &= C R, \end{aligned}$$

the required form (2.1) in this basis is easily concluded. Checking controllability of (2.1) (or observability, since they are implied by each other) gives that  $(A, B)$  is controllable if and only if  $(PQ, P\tilde{B})$  is controllable.  $\square$

A system of the form (2.1) is called *reversible Hamiltonian*. Notice that controllability of  $(PQ, P\tilde{B})$  can be interpreted as controllability of the system

$$P^{-1} \dot{x} = Qx + \tilde{B}u$$

(this observation was made in [2]). This is a gradient system with potential function  $\frac{1}{2}x^T Qx$  and Riemannian metric  $P^{-1}$  (notice that  $\frac{1}{2}\dot{q}^T P^{-1} \dot{q}$  is the kinetic energy) (see for instance [3,9]). We will call

$$P^{-1} \dot{x} = Qx + \tilde{B}u,$$

$$y = \tilde{C}x + Du.$$

the *associated gradient system* of (2.1).

**Theorem 2.** Let  $\{A_i, B_i, C_i, D_i\}$ ,  $i = 1, 2$ , be two minimal reversible Hamiltonian realizations of a same transfer matrix. Then the unique isomorphism  $K$  between  $\{A_1, B_1, C_1, D_1\}$  and  $\{A_2, B_2, C_2, D_2\}$ , i.e.

$$A_2 = K A_1 K^{-1}, \quad B_2 = K B_1, \quad C_2 = C_1 K^{-1},$$

has the form

$$K = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix},$$

$L, M$  ( $n \times n$ )-matrices with  $M = (L^T)^{-1}$  (hence  $K$  is a symplectic matrix). Furthermore let

$$A_i = \begin{pmatrix} 0 & P_i \\ -Q_i & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ \tilde{B}_i \end{pmatrix}$$

and

$$\tilde{C}_i = (C_i \ 0)$$

for  $i = 1, 2$ .

Then

$$K = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix},$$

with  $M = (L^T)^{-1}$ , is an isomorphism between  $\{A_1, B_1, C_1, D_1\}$  and  $\{A_2, B_2, C_2, D_2\}$  if and only if  $L$  is an isomorphism between the two associated gradient systems

$$\{P_1 Q_1, P_1 \tilde{B}_1, \tilde{C}_1, D_1\} \quad \text{and} \quad \{P_2 Q_2, P_2 \tilde{B}_2, \tilde{C}_2, D_2\}.$$

In particular if  $L$  is such an isomorphism, then  $L$  is necessarily an isometry between the metrics  $P_1^{-1}$  and  $P_2^{-1}$ , i.e.  $L^T P_2^{-1} L = P_1^{-1}$ .

**Proof.** Because  $\{A_i, B_i, C_i, D_i\}$ ,  $i = 1, 2$ , are Hamiltonian,  $K$  has to be symplectic, i.e.

$$K^T \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} K = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

(see for the linear case [4, Lemma 2.1]). Writing out  $B_1, A_1 B_1, \dots$  and  $B_2, A_2 B_2, \dots$  and using controllability, one sees (since for  $r = 0, 1, 2, \dots$   $K$  has to satisfy  $K A_i^r B_i = A_i^r B_i$ ) that

$$K = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}.$$

Because  $K$  is symplectic,  $M = (L^T)^{-1}$ .

Now assume

$$K = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}$$

is an isomorphism, then it easily follows that

$$L^T P_2^{-1} L = P_1^{-1}, \quad L^T Q_2 L = Q_1,$$

$$P_2 \tilde{B}_2 = L P_1 \tilde{B}_1, \quad \tilde{C}_2 = \tilde{C}_1 L^{-1}.$$

From the first two equations it follows that

$$P_2 Q_2 = L P_1 Q_1 L^{-1}.$$

Hence  $L$  is an isomorphism between

$$\{P_i Q_i, P_i \tilde{B}_i, \tilde{C}_i, D_i\}, \quad i = 1, 2.$$

Conversely assume  $L$  is an isomorphism, i.e.

$$P_2 Q_2 = L P_1 Q_1 L^{-1}, \quad P_2 \tilde{B}_2 = L P_1 \tilde{B}_1,$$

$$\tilde{C}_2 = \tilde{C}_1 L^{-1}.$$

Then it follows that for  $r = 0, 1, 2, \dots$

$$P_2^{-1} L (P_1 Q_1)^r P_1 \tilde{B}_1 = (L^T)^{-1} P_1^{-1} (P_1 Q_1)^r P_1 \tilde{B}_1.$$

Therefore by controllability

$$P_2^{-1} L = (L^T)^{-1} P_1^{-1} \quad \text{or} \quad L^T P_2^{-1} L = P_1^{-1}.$$

From this last expression and  $P_2 Q_2 = L P_1 Q_1 L^{-1}$  it follows that  $L^T Q_2 L = Q_1$ . Then it is easily concluded that

$$\begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}$$

with  $M = (L^T)^{-1}$  is an isomorphism between

$$\{A_i, B_i, C_i, D_i\}, \quad i = 1, 2. \quad \square$$

### 3. Electrical networks

While the interpretation of Theorem 1 is clear for mechanical systems, we will now investigate its consequences for electrical networks. Our framework will be the same as in [9].

The most simple example of a network which can be described with a Hamiltonian structure is given by a capacitor  $C$  coupled to an inductance  $L$ . Parallel coupling gives

$$\begin{pmatrix} \dot{\varphi}_L \\ \dot{q}_C \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} \varphi_L \\ q_C \end{pmatrix} + \begin{pmatrix} 0 \\ I_e \end{pmatrix},$$

$$y = \varphi_L,$$

with  $\varphi_L$  the magnetic flux on the inductance,  $q_C$  the charge on the capacitor and  $I_e$  the external current.

On the other hand series coupling gives

$$\begin{pmatrix} \dot{q}_C \\ \dot{\varphi}_L \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} q_C \\ \varphi_L \end{pmatrix} + \begin{pmatrix} 0 \\ V_e \end{pmatrix},$$

$$y = q_C,$$

with  $V_e$  the external voltage. Notice that in the first, current-controlled case the natural output in a Hamiltonian framework is the magnetic flux, while in the second, voltage-controlled case the natural output is the charge. Also notice that both systems are already reversible Hamiltonian. More

generally, let  $L$  be a block of interconnected inductances. Then the equations of  $L$  are (see [1])

$$I_L = R\varphi_L, \quad \text{with } R = R^T$$

and  $I_L$  and  $\varphi_L$   $n$ -dimensional. Analogously let  $C$  be a block of capacitors given by the equations

$$V_C = Sq_C, \quad \text{with } S = S^T$$

and  $V_C$  and  $q_C$   $n$ -dimensional. Interconnecting  $L$  and  $C$  by

$$V_L = V_C = V_e; \quad I_C = -I_L + I_e$$

gives

$$\begin{pmatrix} \dot{\varphi}_L \\ \dot{q}_C \end{pmatrix} = \begin{pmatrix} 0 & S \\ -R & 0 \end{pmatrix} \begin{pmatrix} \varphi_L \\ q_C \end{pmatrix} + \begin{pmatrix} 0 \\ I_e \end{pmatrix},$$

$$y = \varphi_L, \quad \text{with } R = R^T \text{ and } S = S^T,$$

i.e. a reversible Hamiltonian system.

It will appear that we can also allow for transformers in our network, and we will roughly speaking show that a network can be given a reversible Hamiltonian structure if and only if it consists of capacitors, inductance and transformers.

**Remark.** We would also like to draw the attention to [7] where it is shown that a (non-linear) LC network has a Hamiltonian structure by proving that it satisfies a variational principle.

Assume now that our electrical network is given by a transfer function  $G(s)$ . Suppose  $G(s)$  is an impedance function, i.e. a function from  $I_e$  to  $V_e$  (the case that  $G(s)$  is an admittance is similar, while for the general hybrid case we will make some comments afterwards). Let  $G(s)$  satisfy the identities  $G(s) = G^T(s)$  and  $G(s) = -G(-s)$ . Then

$$F(s) := \frac{1}{s} G(s)$$

(a function from  $I_e$  to  $\varphi_e$ ) satisfies  $F(s) = F^T(-s)$  and  $F(s) = F(-s)$ .

Therefore by Theorem 1 we can find a minimal realization of  $F(s)$ :

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ B_1 \end{pmatrix} u, \quad y = B_1^T q, \quad (3.1)$$

with  $P = P$ ,  $Q = Q^T$  and  $\det P \neq 0$  (assume  $q$  and  $p$   $n$ -dimensional,  $u$  and  $y$   $m$ -dimensional). Since  $G(s)$  is positive real (see [9]) we may even assume  $P > 0$  and  $Q \geq 0$  (however this is not essential). Write  $P = SS^T$  and apply the symplectic basis transformation

$$\begin{pmatrix} S^{-1} & 0 \\ 0 & S^T \end{pmatrix}$$

to (3.1). Then we obtain (with different  $q$  and  $p$ )

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -S^T Q S & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ S^T B_1 \end{pmatrix} u,$$

$$y = B_1^T S q.$$

Write  $S^T Q S = R D R^T$ , with  $D$  a diagonal matrix with diagonal elements  $d_i$ ,  $i = 1, \dots, n$ , and  $R^T = R^{-1}$ . Applying the symplectic transformation

$$\begin{pmatrix} R^{-1} & 0 \\ 0 & R^T \end{pmatrix}$$

gives

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -D & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ R^T S^T B_1 \end{pmatrix} u,$$

$$y = B_1^T S R q, \quad (3.2)$$

and therefore  $\dot{y} = B_1^T S R p$ .

The system

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -D & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ I_m \end{pmatrix} u,$$

$$y = (0 \quad I_m) q$$

can be realized by  $n$  parallel pairs of exactly one inductance and one capacitor. Each pair consists of a parallel coupled unit capacitor and an inductance  $1/d_i$ ,  $i = 1, \dots, n$  (assume for a moment  $d_i \neq 0$ , see Proposition 3).

The input is the external current  $I_e$  and the output the magnetic flux on (some of) the inductances. Therefore the external voltage  $V_e$  equals

$$V_e = \dot{y} = (0 \quad I_m) p.$$

Finally we define the transformer  $T$  by its transformer equation

$$I_2 = R^T S^T B_1 I_1, \quad V_1 = -B_1^T S R V_2.$$

Then coupling of the above constructed LC network with  $T$  gives exactly the equations (3.2) and hence forms a realization of  $G(s)$ .

However it is possible that this realization is not minimal. In fact we can state:

**Proposition 3.**

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u,$$

$$\dot{y} = (B_1^T \ 0) \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

$$+ (B_1^T \ 0) \begin{pmatrix} 0 \\ B_1 \end{pmatrix} u = B_1^T P p$$

is a minimal realization of  $G(s)$  if and only if  $\det Q \neq 0$ .

**Proof.**

$$\left\{ \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix}, (0 \ B_1^T P) \right\}$$

is observable iff

$$\left\{ \begin{pmatrix} 0 & -Q \\ P & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ P B_1 \end{pmatrix} \right\}$$

is controllable. Hence by Theorem 1 iff  $\{QP, QPB_1\}$  is controllable. If  $\{QP, QPB_1\}$  controllable, then necessarily  $\det Q \neq 0$ . Conversely let  $\det Q \neq 0$ . Then  $\{QP, QPB_1\}$  is controllable iff

$$\{Q^{-1}(QP)Q, Q^{-1}(QPB_1)\} = \{PQ, PB_1\}$$

is controllable. This last statement follows from Theorem 1.  $\square$

When (3.2) is not a minimal realization of  $G(s)$ , then we can find by the last proposition a matrix  $S$  such that

$$S^T Q S = \begin{pmatrix} \tilde{Q} & 0 \\ 0 & 0 \end{pmatrix}$$

with  $\det \tilde{Q} \neq 0$ . Applying the symplectic transformation

$$\begin{pmatrix} S^{-1} & 0 \\ 0 & S^T \end{pmatrix}$$

to (3.2) gives

$$\begin{cases} \dot{q} = \tilde{P} p, \\ \dot{p} = - \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} q + \tilde{B}_1 u, \quad y = \tilde{B}_1^T q. \end{cases}$$

Writing

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad \text{and} \quad \tilde{P} = \begin{pmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{pmatrix}$$

corresponding to

$$\begin{pmatrix} \tilde{Q} & 0 \\ 0 & 0 \end{pmatrix}$$

we see that

$$\begin{cases} \dot{q}_1 = \tilde{P}_1 p, \\ \dot{p} = \begin{pmatrix} -Q \\ 0 \end{pmatrix} q + \tilde{B}_1 u, \quad \dot{y} = B_1^T \tilde{P} p, \end{cases} \quad (3.3)$$

is a minimal realization of  $G(s)$ .

Notice that we may interpret this reduction as deleting some capacitors or inductances (depending on whether  $G(s)$  is an impedance or admittance function).

We sum up these results in:

**Theorem 4.** Let  $G(s)$  be a transfer function of an electrical network satisfying  $G(s) = G^T(s)$  and  $G(s) = -G(-s)$ . Then we can construct a minimal realization with only capacitors, inductances and transformers. Moreover if the McMillan degree of  $s^{-1}G(s)$  is equal to the McMillan degree of  $G(s)$ , the capacitors and inductance can be arranged in pairs.

**Remark.** If  $G(s)$  is a hybrid representation we can state a similar result by using the following generalization of Theorem 1. Let  $F(s)$  satisfy

$$\Sigma_e F(s) = F^T(-s) \Sigma_e \quad \text{and} \quad \Sigma_e F(s) = F(-s) \Sigma_e$$

( $\Sigma_e$  denoting a signature matrix). Then we can construct a minimal realization  $\{A, B, C, D\}$  with  $A$  as in Theorem 1 and

$$B = -RB\Sigma_e, \quad \Sigma_e C = CR, \quad \Sigma_e D = D\Sigma_e$$

and

$$\Sigma_e B^T J = C, \quad \Sigma_e D = D^T \Sigma_e.$$

We now want to go the other way around and assume that we have an electrical network composed of capacitors, inductances and transformers. We do reactance extraction and write (assume the

network is voltage controlled)

$$\begin{pmatrix} I_C \\ V_L \\ I_e \end{pmatrix} = \begin{pmatrix} 0 & Z_1 & 0 \\ -Z_1^T & 0 & -Z_2 \\ 0 & Z_2^T & 0 \end{pmatrix} \begin{pmatrix} V_C \\ I_L \\ V_e \end{pmatrix}.$$

Following [9], this constitutes a state space description  $\{A, B, C, D\}$  with

$$A = \begin{pmatrix} 0 & -Z_1 \\ Z_1^T & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ Z_2 \end{pmatrix},$$

$$C = (0 \quad Z_2^T), \quad D = 0.$$

From the controllability of  $\{A, B\}$  it follows that  $Z_1$  is surjective, or  $Z_1^T$  injective. Assume first that  $Z_1$  is square. Then by applying the state space transformation

$$\begin{pmatrix} -Z_1^{-1} & 0 \\ 0 & I \end{pmatrix}$$

we obtain

$$\tilde{A} = \begin{pmatrix} 0 & I \\ -Z_1^T Z_1 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ Z_2 \end{pmatrix},$$

$$\tilde{C} = (0 \quad Z_2).$$

Integrating  $y$  gives ( $\dot{z} = y$ )

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -Z_1^T Z_1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ Z_2 \end{pmatrix} u,$$

$$z = Z_2^T q,$$

i.e. a time-reversible Hamiltonian system.

Consider now the case that  $Z_1$  is not square but has more columns than rows (because  $Z_1$  is surjective). We will briefly sketch how this boils down to the same situation as in equation (3.3). Let  $Z_1$  be a  $(k \times p)$  matrix ( $k \leq p$ ). Then construct a non-singular  $p \times p$  matrix  $S$  such that

$$SZ_1^T = \begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

Applying the transformation

$$\begin{pmatrix} I_{2n-k} & 0 \\ 0 & -S \end{pmatrix}$$

gives

$$\tilde{A} = \begin{pmatrix} 0 & -Z_1 S^{-1} \\ -I_k & 0 \\ 0 & \end{pmatrix}.$$

Denoting  $\tilde{Q} = I_k$ ,  $\tilde{P}_1 = -Z_1 S^{-1}$  (it is easy to see that we can construct  $S$  in such a way that  $-Z_1 S^{-1}$  can be completed to a symmetric matrix), we see that we have obtained equation (3.3).

#### 4. Conclusion

We have shown that the notions of a Hamiltonian and a reversible system are compatible with each other, resulting in a natural subclass of Hamiltonian systems. Furthermore to this subclass can be associated the class of gradient systems. A generalization of the results to the nonlinear case, using the concepts and techniques as indicated in [5,6] and the references cited there, is being investigated.

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